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Renate Tobies

Felix Klein

Visions for Mathematics, Applications,
and Education

Revised by the Author and Translated by Valentine A. Pakis



Figure 1: Felix Klein, 1875 [Hillebrand].

**“Whoever shall live on in the memory of the wide world
must have had an impact on that world.”**

(BLUMENTHAL 1928, p. 2)

PREFACE

Richard Courant spoke euphorically about Felix Klein (1849-1925): “His life was full of intellectual vigor and the will to act, both spurred by a brilliant imagination that was always contriving more and more new designs. He was entirely the sort of wise man and ruler described in Plato’s *Republic*.”¹ With his *Erlangen Program*, Klein convincingly redefined geometry: geometric properties as invariants of transformation groups. He systematized mathematical theories by recognizing and explaining the interrelations between different disciplines. His visionary programs concerned mathematics and its applications, but also history, philosophy, and pedagogy from kindergarten through higher education. He was extraordinarily engaged, as his admirers would say, in raising awareness for the “eminent cultural significance of mathematics and its applications.”²

In 1892, the famous Austrian theoretical physicist Ludwig Boltzmann extolled Klein’s all-encompassing activity:

[...] Klein’s work encompasses almost all areas of mathematics. Especially noteworthy are his contributions to the following areas:

- 1 Algebra and its application to the theory of algebraic forms, number theory, geometry, the resolution of higher equations.
- 2 General theory of functions, theory of elliptic, Abelian, θ -functions and of Riemann surfaces;
- 3 Theory of differential equations;
- 4 Foundations of geometry, curvature and other shape relations of curves and surfaces, also newer geometry and projectivity, the application of geometry to mechanics.³

The present book deals with Klein’s multifaceted programs and the development of his works. It sheds light on how Klein became a scientist who was able to attract students – male and female alike – to follow his visions.

In 1870, Klein became the first German mathematician to seek personal contact with French mathematicians since Plücker, Dirichlet, and Jacobi had done this some decades before. Klein traveled several times to the British Isles, to Italy, to the United States, etc. He was at the center of the first international congresses of mathematicians and was elected the first chairman of the International Commission on Mathematical Instruction (ICMI) in 1908. In Germany, Felix Klein steered the fortunes of the German Mathematical Society three times as its chairman and, as a professor emeritus, he was still considered the “foreign minister” of mathematics. In the 1890s, the French mathematician Charles Hermite gushingly

1 COURANT 1926, p. 211.

2 [UAG] Math.-Nat. Fak. 25, Valentiner (report from July 19, 1924).

3 Quoted from HÖFLECHNER 1994, pp. 173–74 (Boltzmann to Paul von Groth). – Regarding the context, see Section 6.5.2.

referred to Klein as “a new Joshua in the promised land.”⁴ Klein became a citizen of the world, explicitly condemning national chauvinism (see Section 8.4).

A precocious student, Klein had completed secondary school at the age of sixteen, earned a doctoral degree at the age of nineteen, and completed his post-doctorate (*Habilitation*) at the age of twenty-one. He was offered his first full professorship at the age of twenty-three, at the University of Erlangen (1872). This was followed by positions at the Polytechnikum in Munich (1875), the University of Leipzig (1880), and the University of Göttingen (as of 1886).

More than just focusing on Klein’s professional achievements, this book will also be concerned with Klein as a person. At the age of twenty-six, he married Anna Hegel, the granddaughter of the great philosopher. Her extant letters to Felix Klein document their good relationship and demonstrate that she was often involved in his academic work. Of their four children (one son, three daughters), their son would go on to pursue a technical career. Their youngest daughter studied mathematics, physics, and English in Göttingen and at Bryn Mawr College in the United States. She achieved a distinguished career as a teacher and school principal until 1932; later, she was demoted during the Nazi regime.

Klein cultivated a cooperative working style. At the age of twenty, he found his most important partner in the Norwegian Sophus Lie. Klein wanted to work together, not in competition. Nevertheless, he had to deal with opponents, competitors, different views and interests. David Hilbert, who, on the occasion of Klein’s sixtieth birthday in 1909 also invited Henri Poincaré and Gösta Mittag-Leffler to Göttingen, referred in his speech then to Klein’s opponents and supporters and expressed his own affinity for Klein.⁵

Klein was not, from the outset, the “Zeus enthroned above the other Olympians,” as Max Born experienced him during his own years as a student (“He was known among us as ‘the Great Felix’,” Born went on, “and he controlled our destinies”).⁶ We will instead encounter a mathematician who was often plagued by self-doubt and who worried that he might not be able to live up to his own high standards. Early translations of his work and his efforts as the chief editor of the journal *Mathematische Annalen* brought him fame and influence.

With his finger on the pulse of international trends, Klein left a lasting mark on many areas of mathematics, its applications, and organization in Germany. In an astounding number of areas, he was in fact a pioneer.⁷ At the University of Göttingen, Klein had laid the foundation for a new golden era and had pointed the way ahead, as Hilbert put it (see Appendix 12). This meant that he appointed the best scientists (among them Hilbert, Hermann Minkowski, Carl Runge, Ludwig Prandtl, Edmund Landau) to work beside him, that he found new ways to retain them in Göttingen, and that he established new institutes by acquiring funds from industry – inspired by the example of American universities.

4 For the context of this quotation, see Section 8.2.2 of this book.

5 See TOBIES 2019b, pp. 513–14, Engl. trans. in ROWE 2018a, pp. 198–99.

6 BORN/BORN 1969, p. 16.

7 For a summary of Klein’s pioneering achievements, see Section 10.2.

Right up into old age, Klein was open to new mathematical, scientific, and technical theories. Thus he also identified open problems in the fields of fluid dynamics and statics. In partial collaboration with Emmy Noether, he made significant contributions to the theory of relativity, acknowledged by Albert Einstein.

Klein recognized the specific talents of his students with great foresight. He promoted gifted persons regardless of their religion, nationality, and gender. He guided more than fifty doctoral students, including two women (an Englishwoman and an American) as well as further students from abroad, to new results.

During his lifetime, he received numerous honors, and his versatility is still widely recognized today. Since the year 2000, the European Mathematical Society has awarded a “Felix Klein Prize” to young scientists for outstanding research in applied mathematics (this award was initiated by the Fraunhofer Institute for Industrial Mathematics in Kaiserslautern). Since 2003, moreover, the ICMI has presented a “Felix Klein Award” for lifetime achievements in the field of mathematical pedagogy. In Germany, several institutions have been named after him. There is a Felix Klein Lecture Hall and a Felix Klein Colloquium at the Heinrich Heine University in Düsseldorf (Klein’s birthplace) and at the University of Leipzig as well. There is a Felix Klein building at the University of Erlangen and a Felix Klein program at the Technische University in Munich (including a “Felix Klein Teaching Prize”). In Göttingen, there is a secondary school named after Felix Klein, and the meeting room of the Mathematical Institute of the University is adorned by the original Max Liebermann portrait of Klein. The names of the donors who funded this painting are an expression of Klein’s worldwide network, which extended as far as India and Japan.⁸

After the late Leipzig historian of mathematics Hans Wußing had encouraged me to study the life and work of Felix Klein, it was the American historian of mathematics David E. Rowe who first enabled me – when Germany was still divided – to study the archival materials pertaining to Klein in Göttingen. The mathematician Helmut Neunzert invited me to give lectures at the University of Kaiserslautern with the following words: “We like to use Klein’s arguments to promote the applications of mathematics even today!” A Felix Klein Center was established there in 2008.

Robert Fricke, the mathematician and erstwhile rector of the Technische Hochschule in Braunschweig, aptly compared Felix Klein (an uncle of Fricke’s wife) to a triptych, the central panel of which should be devoted to Klein the researcher, while the two flanking panels should depict him as an academic teacher and an outstanding organizer.⁹ The goal of this book is to put this triptych into words and enrich it with a human dimension.

Jena, March of 2021

Renate Tobies

8 See Section 8.5.2, and Appendix 10, Fig. 43. – The portrait of Hilbert in the same room was painted in 1928 by Eugen Spiro, who was forced to emigrate in 1935. On Hilbert, see in particular Sections 6.3.7.3 and 7.9 in this book.

9 FRICKE 1919, p. 275. – See the genealogy in [Figure 2](#).

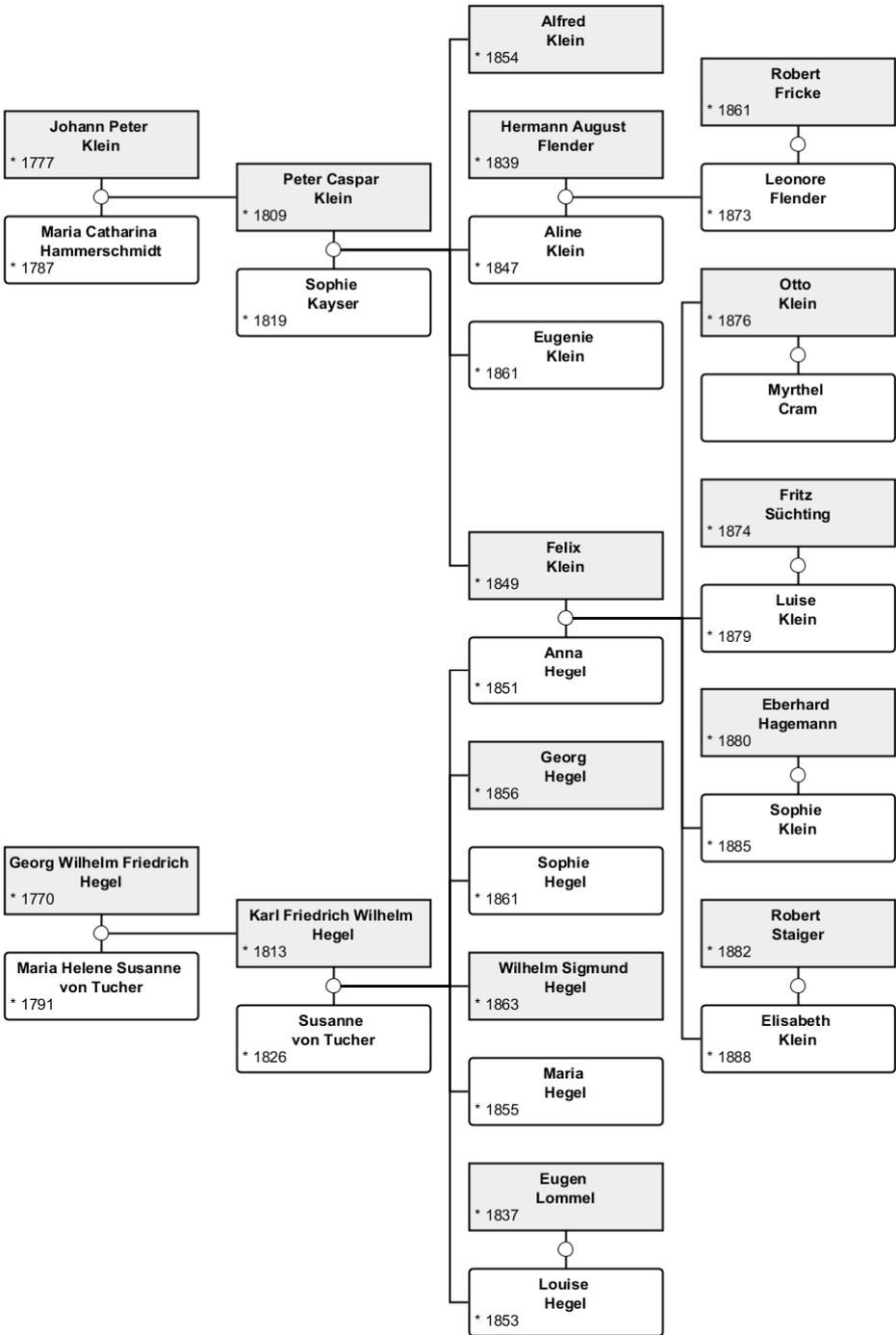


Figure 2: An excerpt from the Klein-Hegel Family Tree (my own design, from [Hillebrand])

CONTENTS

Preface	vii
List of Tables	xviii
List of Figures	xviii
1 Introduction	1
1.1 The State of Research	3
1.2 Guiding Questions.....	8
1.3 Editorial Remarks	13
2 Formative Groups	17
2.1 The Klein–Kayser Family	17
2.1.1 A Royalist and Frugal Westphalian Upbringing.....	17
2.1.2 Talent in School and Wide Interests as Gifts from His Mother’s Side.....	20
2.1.3 Felix Klein and His Siblings	21
2.2 School Years in Düsseldorf.....	22
2.2.1 Earning His <i>Abitur</i> from a Gymnasium at the Age of Sixteen	23
2.2.2 Examination Questions in Mathematics	25
2.2.3 Interests in Natural Science During His School Years	26
2.3 Studies and Doctorate in Bonn.....	28
2.3.1 Coursework and Seminar Awards.....	29
2.3.2 Assistantship and a Reward for Winning a Physics Contest	34
2.3.3 Assisting Julius Plücker’s Research in Geometry.....	36
2.3.4 Doctoral Procedure	40
2.4 Joining Alfred Clebsch’s Thought Community	45
2.4.1 The Clebsch School	47
2.4.2 The Journal <i>Mathematische Annalen</i>	53
2.4.3 Articles on Line Geometry, 1869.....	58
2.5 Broadening His Horizons in Berlin.....	61
2.5.1 The Professors in Berlin and Felix Klein.....	62
2.5.2 Acquaintances from the Mathematical Union: Kiepert, Lie, Stolz..	66
2.5.3 Cayley’s Metric and Klein’s Non-Euclidean Interpretation	71
2.6 In Paris with Sophus Lie	73
2.6.1 Felix Klein and French Mathematicians	74
2.6.2 Collaborative Work with Sophus Lie.....	78
2.6.2.1 Notes on W-Configurations	78

2.6.2.2 Principal Tangent Curves of the Kummer Surface	80
2.6.3 A Report on Mathematics in Paris	82
2.7 The Franco-Prussian War and Klein's <i>Habilitation</i>	83
2.7.1 Wartime Service as a Paramedic and Its Effects	84
2.7.2 <i>Habilitation</i>	88
2.8 Time as a <i>Privatdozent</i> in Göttingen	90
2.8.1 Klein's Teaching Activity and Its Context	91
2.8.2 An Overview of Klein's Research Results as a <i>Privatdozent</i>	98
2.8.3 Discussion Groups	110
2.8.3.1 A Three-Man Club with Clebsch and Riecke	110
2.8.3.2 The Mathematical and Natural-Scientific Student Union	113
2.8.3.3 A Scientific Circle: Eskimo	115
2.8.3.4 The "Social Activity" of Bringing Mathematicians Together	117
3 A Professorship at the University of Erlangen	123
3.1 Research Trends and Doctoral Students	125
3.1.1 The Vision of the <i>Erlangen Program</i>	126
3.1.2 Klein's Students in Erlangen	132
3.1.3 New Research Trends	138
3.1.3.1 On a New Type of Riemann Surface	139
3.1.3.2 The Theory of Equations	143
3.2 Inaugural Lecture: A Plan For Mathematical Education	144
3.3 First Trip to Great Britain, 1873	147
3.4 Trips to Italy	153
3.5 Developing the Mathematical Institution	158
3.6 Family Matters	160
3.6.1 His Friends Marry and Klein Follows Suit	161
3.6.2 Klein's Father-in-Law, the Historian Karl Hegel	164
3.6.3 Anna Hegel, Felix Klein, and Their Family	166
4 A Professorship at the Polytechnikum in Munich	171
4.1 A New Institute and New Teaching Activity	173
4.1.1 Creating a Mathematical Institute	174
4.1.2 Reorganizing the Curriculum	176
4.2 Developing His Mathematical Individuality	178
4.2.1 The Icosahedron Equation	179
4.2.2 Number Theory	183
4.2.3 Elliptic Modular Functions	184
4.2.4 Klein's Circle of Students in Munich	191
4.2.4.1 Phase I: 1875–1876	191
4.2.4.2 Phase II: 1876–1880	193
4.3 Discussion Groups in Munich	201

4.3.1 A Mathematical Discussion Group with Engineers and Natural Scientists	201
4.3.2 The Mathematical Student Union and the Mathematical Society ..	204
4.3.3 The Meeting of Natural Scientists in Munich, 1877	205
4.4 “Ready Again for a University in a Small City”	208
5 A Professorship for Geometry in Leipzig	213
5.1 Klein’s Start in Leipzig and His Inaugural Address	215
5.2 Creating a New Mathematical Institution	218
5.3 Teaching Program	221
5.3.1 Lectures: Organization, Reorientation, and Deviation from the Plan	221
5.3.2 The Mathematical Colloquium / Exercises / Seminar	227
5.4 The Kleinian “Flock”	232
5.4.1 Post-Doctoral Mathematicians	233
5.4.2 Klein’s Foreign Students in Leipzig	243
5.4.2.1 The First Frenchman and the First Briton	244
5.4.2.2 The First Americans	245
5.4.2.3 The Italians	246
5.4.2.4 Mathematicians from Switzerland and Austria-Hungary	248
5.4.2.5 Russian and Other Eastern European Contacts	250
5.5 Fields of Research	252
5.5.1 Mathematical Physics / Physical Mathematics	253
5.5.1.1 Lamé’s Function, Potential Theory, and Carl Neumann	253
5.5.1.2 On Riemann’s Theory of Algebraic Functions and Their Integrals	255
5.5.2 Looking Toward Berlin	260
5.5.2.1 Gathering Sources	260
5.5.2.2 The Dirichlet Principle	261
5.5.2.3 Klein’s Seminar on the Theory of Abelian Functions (1882)	264
5.5.2.4 Openness vs. Partiality	266
5.5.3 Looking Toward France	267
5.5.3.1 French Contributors to <i>Mathematische Annalen</i>	267
5.5.3.2 Klein’s Correspondence with Poincaré	269
5.5.4 Three Fundamental Theorems	272
5.5.4.1 The Loop-Cut Theorem (Rückkehrschnitttheorem)	273
5.5.4.2 Theorem of the Limit-Circle (Grenzkreistheorem)	273
5.5.4.3 The (General) Fundamental Theorem	277
5.5.4.4 Remarks on the Proofs	279
5.5.5 The Polemic about and with Lazarus Fuchs	282
5.5.6 The Icosahedron Book	286
5.5.7 A Book on the Theory of Elliptic Modular Functions	291

5.5.7.1 Supplementing the Theory	291
5.5.7.2 Who Should Be the Editor? – Georg Pick	294
5.5.8 Hyperelliptic and Abelian Functions	298
5.6 Felix Klein and Alfred Ackermann-Teubner	300
5.7 Felix Klein in Leipzig’s Intellectual Communities	307
5.7.1 A Mathematicians’ Circle	308
5.7.2 The <i>Societas Jablonoviana</i>	308
5.7.3 The Royal Saxon Society of Sciences in Leipzig	310
5.8 Turning His Back on Leipzig	314
5.8.1 Weighing Offers from Oxford and Johns Hopkins	314
5.8.2 The Physicist Eduard Riecke Arranges Klein’s Move to Göttingen	316
5.8.3 The Appointment of Sophus Lie as Klein’s Successor – and the Reactions	320
6 The Start of Klein’s Professorship in Göttingen, 1886–1892	325
6.1 Family Considerations	326
6.2 Dealing with Colleagues, Teaching, and Curriculum Planning	328
6.2.1 The Relationship Between Klein and Schwarz	328
6.2.2 The Göttingen <i>Privatdozenten</i> Hölder and Schoenflies	329
6.2.3 Klein’s Teaching in Context	332
6.3 Independent and Collaborative Research	337
6.3.1 The Theory of Finite Groups of Linear Substitutions: The Theory of Solving Equations of Higher Degree	337
6.3.2 Hyperelliptic and Abelian Functions	339
6.3.3 The Theory of Elliptic Modular Functions (Monograph)	341
6.3.4 The Theory of Automorphic Functions (Monograph)	343
6.3.5 The Theory of Lamé Functions and Potential Theory	344
6.3.6 Refreshing His Work on Geometry	347
6.3.7 Visions: Internationality, Crystallography, Hilbert’s Invariant Theory	352
6.3.7.1 An Eye on Developments Abroad	352
6.3.7.2 Arthur Schoenflies and Crystallography	356
6.3.7.3 Felix Klein and Hilbert’s Invariant Theory	357
6.4 Bringing People and Institutions Together	361
6.4.1 The <i>Professorium</i> in Göttingen	361
6.4.2 A Proposal to Relocate the Technische Hochschule in Hanover to Göttingen	362
6.4.3 The Idea of Reorganizing the Göttingen Society of Sciences	364
6.4.4 Felix Klein and the Founding of the German Mathematical Society	367
6.5 The Pivotal Year of 1892	373
6.5.1 Refilling Vacant Professorships in Prussia	373

6.5.1.1 Berlin, Breslau, and Klein's System for Classifying Styles of Thought	373
6.5.1.2 Hiring a Successor for H.A. Schwarz in Göttingen	377
6.5.2 A Job Offer from the University of Munich and the Consequences	379
7 Setting the Course, 1892/93–1895	383
7.1 Klein's Assistants and His Principles for Choosing Them	385
7.2 The Göttingen Mathematical Society	392
7.3 Turning to Secondary School Teachers	397
7.4 A Trip to the United States	401
7.4.1 The World's Fair in Chicago and the Mathematical Congress	401
7.4.2 Twelve Lectures by Klein: The Evanston Colloquium	404
7.4.3 Traveling from University to University	406
7.4.4 Repercussions	407
7.5 The Beginnings of Women Studying Mathematics	411
7.6 Actuarial Mathematics as a Course of Study	418
7.7 Contacting Engineers and Industrialists	421
7.8 The <i>Encyklopädie</i> Project	425
7.9 Klein Succeeds in Hiring David Hilbert	434
8 The Fruits of Klein's Efforts, 1895–1913	437
8.1 A Center for Mathematics, Natural Sciences, and Technology	438
8.1.1 The Göttingen Association	439
8.1.2 Applied Mathematics in the New Examination Regulations and the Consequences	445
8.1.3 Aeronautical Research	450
8.2 Maintaining His Scientific Reputation	454
8.2.1 Automorphic Functions (Monograph)	455
8.2.2 Geometric Number Theory	457
8.2.3 A Monograph on the Theory of the Spinning Top	461
8.2.4 Inspiring Ideas in the Fields of Mathematical Physics and Technology	465
8.2.4.1 Hydrodynamics / Hydraulics	466
8.2.4.2 Statics	468
8.2.4.3 The Theory of Friction	471
8.2.4.4 The Special Theory of Relativity	472
8.3 Program: The History, Philosophy, Psychology, and Instruction of Mathematics	474
8.3.1 The History of Mathematics	477
8.3.2 Philosophical Aspects	481
8.3.3 Psychological-Epistemological Classifications	490
8.3.4 The "Kleinian" Educational Reform	493
8.3.4.1 Suggestions for Reform	500

8.3.4.2 A Polemic about the Teaching of Analysis at the University	508
8.4 International Scientific Cooperation	510
8.5 Early Retirement and Honors	514
8.5.1 Recovering and Working in the Hahnenklee Sanatorium	515
8.5.2 Max Liebermann's Portrait of Felix Klein	519
8.5.3 The Successors to Klein's Professorship	522
9 The First World War and the Postwar Period.....	525
9.1 Political Activity During the First World War	526
9.1.1 The Vows of Allegiance of German Professors to Militarism	527
9.1.2 A Plea for Studying Abroad	531
9.2 History of Mathematics, the "Cry for Help of Modern Physics," and Edition Projects	534
9.2.1 Remarks on Klein's Historical Lectures	536
9.2.2 Felix Klein and the General Theory of Relativity	538
9.2.3 The Golden Anniversary of Klein's Doctorate, and Edition Projects	545
9.3 Mathematical Education – International and National	548
9.3.1 The International Commission on Mathematical Instruction	549
9.3.2 Countering the Restriction of Mathematics and the Natural Sciences	551
9.4 Support for Research	556
9.4.1 The Emergency Association of German Science	557
9.4.2 The Gauss-Weber / Helmholtz Society	560
9.5 End of Life	564
10 Concluding Remarks	569
10.1 A Summary of Findings	570
<i>On the Continuity of Klein's Field of Research</i>	570
<i>Creating Favorable Conditions for Good Scientific Work</i>	572
<i>Focusing on Problems of Mathematical Instruction at Schools</i>	573
<i>Klein's Handling of His Health Problems</i>	573
A Summary of the Aspects that Guided the Research for the Present Biography	574
10.2 A Pioneer	585

Appendix: A Selection of Documents.....	593
1) A letter from Felix Klein to Heinrich von Mühler, the Prussian Minister of Religious, Educational, and Medical Affairs (Minister of Culture).	593
2) An application submitted by Felix Klein to the Academic Senate of the University of Erlangen for funding to improve the collection of the University Library's mathematical section (November 15, 1872).	594
3) Nomination of Dr. Felix Klein, full professor of mathematics at the Technische Hochschule in Munich, to be made an extraordinary member of the mathematical-physical class of the Royal Bavarian Academy of Sciences, June 7, 1879.....	597
4) A report by the Philosophical Faculty at the University of Göttingen concerning its decision to propose Felix Klein as the successor to Moritz Abraham Stern, along with separate opinions by the professors Ernst Schering and Hermann Amandus Schwarz (January 1885).	598
5) On the scientific polemic between Felix Klein and Lazarus Fuchs. An excerpt of a letter (in draft form) from Felix Klein to Wilhelm Förster (a professor of astronomy at the University of Berlin), January 15, 1892.	603
6) Letters concerning the potential successor to H.A. Schwarz's full professorship at the University of Göttingen.	605
7) Felix Klein on the draft of Ludwig Bieberbach's dissertation, which was supervised by the <i>Privatdozent</i> Paul Koebe at the University of Göttingen.....	607
8) Dr. Klaus, a neurologist at the Sanatorium for Neurology and Internal Medicine in Hahnenklee: two reports on the state of Felix Klein's health.	608
9) Nomination of Felix Klein to be made a corresponding member of the Royal Prussian Academy of Sciences in Berlin, February 27, 1913.	609
10) Speeches given on May 25, 1913 upon the presentation of Max Liebermann's portrait to Felix Klein.	611
11) Virgil Snyder from Ithaca (New York) to Felix Klein, a letter, dated July 4, 1924, concerning the International Congress of Mathematicians in Toronto, Canada from 11 August to 16 August 1924.	618
12) David Hilbert's eulogy for Felix Klein, delivered at the session of the Göttingen Mathematical Society held on June 23, 1925, one day after Klein's death.	621
Bibliography	623
Index of Names.....	655

LIST OF TABLES

Table Number	Title	Page
1	Evaluations of Felix Klein's achievements from his <i>Abitur</i> diploma (August 3, 1865).	24
2	Examination questions in mathematics (1865).	26
3	Courses attended by Felix Klein at the University of Bonn (1865–1868).	31
4	A list of course offerings in mathematics, physics, and astronomy at the University of Göttingen for the summer semester of 1871.	93
5	On the <i>Erlangen Program</i> .	127
6	Participants in Klein's Research Seminars, 1880/81–1885/86.	230
7	Lectures at the Göttingen Mathematical Society, 1892/93.	392
8	Applied Mathematics in the Prussian Examination Regulation for Teaching Candidates at Secondary Schools as of 1898.	446
9	Members of the Commission for Education in the Upper House (<i>Herrenhaus</i>) of the Prussian Parliament, formed on March 19, 1909.	500
10	Felix Klein's Courses and Other Activity, 1914–1922.	535

LIST OF FIGURES

Figure Number	Title	Page
1	Felix Klein, 1875.	v
2	An excerpt from the Klein – Hegel Family Tree.	x
3	An excerpt of a letter from Felix Klein to Sophus Lie dated April 1, 1872.	16
4	Felix Klein at the age of two (unknown illustrator).	18
5	Felix Klein's Doctoral Certificate, December 12, 1868.	43
6	Alfred Clebsch.	46
7	The title page of volume 6 of <i>Mathematische Annalen</i> (1873).	55
8	A Kummer surface with 16 real nodes.	60
9	A title page of the <i>Bulletin des Sciences Mathématiques et Astronomiques</i> .	76
10	An excerpt of a letter from Klein to Lie dated July 29, 1870, including a sketch of the asymptotic curves between two double points on a Kummer surface.	85
11	Clebsch's diagonal surface, the first model of a cubic surface on which all of its 27 lines are real.	106
12	An illustration of a cubic surface with four real nodes.	109
13	Eduard Riecke.	112
14	The title page of Klein's <i>Erlangen Program</i> (October 1872).	122

15	Klein's circle in Erlangen, 1873. Felix Klein (on the right) with Ferdinand Lindemann, Wilhelm Bretschneider, Siegmund Günther, Adolf Weiler, and Ludwig Wedekind.	134
16	Charles Xavier Thomas's arithmometer. Serial No. 759, built in 1868; dimensions (mm): 460 long, 180 wide, 93 high.	146
17	Felix Klein's certification as a foreign member of the London Mathematical Society, 1875, and the De Morgan Medal, which he became the fourth mathematician (after Cayley, Sylvester, and Rayleigh) to receive in 1893.	152
18	Anna Hegel and Felix Klein's engagement announcement – January 9, 1875.	162
19	A photograph from Anna and Felix Klein's silver wedding anniversary – Sunday, August 19, 1900.	167
20	Klein's modular figure, derived from Dedekind.	185
21	Klein's "main figure" (Hauptfigur) with 2×168 circular arc triangles.	187
22	Adolf Hurwitz.	197
23	Carl Linde, 1872.	202
24	The Klein bottle.	257
25	The title page of Klein's book <i>On Riemann's Theory of Algebraic Functions and Their Integrals</i> .	259
26	An excerpt of Klein's drafted letter to A. Ackermann-Teubner, December 31, 1899.	304
27	Felix Klein's home in Göttingen, Wilhelm-Weber-Straße 3.	326
28	Rohns Tavern on the Hainberg.	327
29	The founding members of the German Mathematical Society (Deutsche Mathematiker-Vereinigung, DMV), September 18, 1890.	370
30	Felix Klein's certification as a foreign member of the Società Italiana delle Scienze, 1896.	382
31	The Göttingen Mathematical Society, 1902.	394
32	Grace Chisholm and Luise Klein.	414
33	An <i>ENCYKLOPÄDIE</i> trip to Wales. Felix Klein (seated in the middle) and Arnold Sommerfeld (left) with George Hartley Bryan (standing in the middle) and Bryan's family.	432
34	The Göttingen Association for the Promotion of Applied Physics and Mathematics. An invitation to the celebration of its tenth anniversary, February 22, 1908.	441
35	The title page of the French edition of Riemann's Collected Works (1898), including Felix Klein's speech (discours) on Riemann, KLEIN 1894.	464
36	Klein's updated plan for the volume <i>Die mathematische Wissenschaften</i> of the project <i>Die Kultur der Gegenwart</i> [The Culture of the Present], August 1912.	476
37	The title page of the first issue of the journal <i>L'Enseignement mathématique</i> (1899), which has been the journal of the ICMI since 1908.	495
38	Committees (etc.) in which Klein discussed educational issues.	497
39	Bockswiese-Hahnenklee in the Harz mountains, a view of the sanatorium.	515
40	Max Liebermann's portrait of Felix Klein (1912).	521

41	Felix Klein, a drawing by Leonard Nelson.	568
42	Felix Klein's diploma for his honorary doctorate from the Jagiellonian University of Krakow (Uniwersytet Jagiellonski w Krakowie), 1900.	592
43	A list of donors who sponsored Max Liebermann's painting of Klein's portrait in 1912.	614
44	The certification of Felix Klein's election as a foreign associate of the National Academy of Sciences of the United States of America, April 21, 1898.	620
45	Felix and Anna Klein's gravestone in Göttingen's old city cemetery.	622
46	Felix Klein's diploma for his honorary doctorate (doctoris rerum politicarum dignitatem et ornamenta) from the University of Berlin, April 25, 1924.	654